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A numerical study of convective and radiative transfer in ducts of rectangular and equilateral cross sections

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1. INTRODUCTION

THE TOPIC of laminar forced convection in ducts having simple and non-simple cross sections has been extensively studied [1, 2]. A recent review on the subject [3] covered a wide range of solution methods with some applications. Many of the investigations cited in this review have not accounted for the simultaneous effect of forced convection and radiation in the thermal entrance region of ducts where the flowing medium is a participating gas. However, other investigations [4–7] specializing on circular tubes have clearly demonstrated the important influence of radiation on the forced convection process in high temperature gas flows.

The present investigation addresses the problem of combined laminar convection and radiation transfer in the region of thermal development of non-circular isothermal ducts. Under the assumption of a gray gas [8] and fully developed velocity at the entrance, the thermal development of the participating medium in an isothermal duct accounts for both emission and absorption. Rigorous modeling of radiation in a participating medium formulation involves the solution of a non-linear integral equation. However, various approximates of accurate differential methods, such as the method of moments [8], model the radiation transfer by an elliptic partial differential equation which describes the irradiation distribution in the medium.

The developing temperature is determined by solving the three-dimensional energy equation, via the method of lines (MOL) [9]. The transversal derivatives in this parabolic partial differential equation are replaced by finite difference formulations while the axial derivative remains continuous. The integration domain is divided into a collection of lines parallel to the axial coordinate of the duct. Thus, the partial differential energy equation is replaced by a system of non-linear ordinary differential equations of the first order, where the dependent variables are the temperatures along each line and the independent variable is the axial coordinate. For computational purposes, the retention of equal transversal intervals in the presence of irregular boundaries presents a complication requiring special attention for nodes in the neighborhood of the boundary [10]. The construction of the grid in the duct cross section is done such that the dividing lines in the grid itself coincide with the irregular boundaries.

2. GOVERNING EQUATIONS

Consider a fully developed laminar gas flow inside a straight duct of non-circular cross section. At $x = 0$, the gas temperature is uniform and equal to T_e, and for $x > 0$, the outer surface of the duct is maintained at an isothermal temperature, T_w. The participating gas in the region of thermal development is assumed to be gray, emitting and absorbing. Under the idealization of temperature-invariant properties, the corresponding energy equation may be expressed by

$$u\frac{\partial T}{\partial x} = \alpha\left(\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)-\frac{1}{\rho c_p}\operatorname{div}\mathbf{q}_R \qquad (1)$$

where x, y and z are the Cartesian coordinates of the duct.

Turning attention to the second term on the right-hand side of equation (1), div $\mathbf{q}_R$ characterizes the radiative contribution of a participating gas. In fact, this contribution may be modeled by an approximate differential method, such as the method of moments in two dimensions [8]. Accordingly, the radiative transfer equation (RTE) is conveniently expressed in differential form as follows:

$$\operatorname{div}\mathbf{q}_R = -K_a(G-4\sigma T^4) \qquad (2)$$

where K_a is the total volumetric absorption coefficient, σ the Stefan–Boltzmann constant and the irradiation G is given by the equation

$$\frac{\partial^2 G}{\partial x^2}+\frac{\partial^2 G}{\partial y^2}+\frac{\partial^2 G}{\partial z^2} = 3\beta K_a(G-4\sigma T^4). \qquad (3)$$

Upon introduction of a reference temperature $T_{ref} = T_w$ and a characteristic length b, the customary dimensionless variables and parameters are

$$U=\frac{u}{u_m},\quad \phi=\frac{T}{T_w},\quad \frac{T-T_w}{T_e-T_w}=\frac{\phi-1}{\phi_e-1}$$

$$Y=\frac{y}{b},\quad Z=\frac{z}{b},\quad X=\frac{x}{D_h\,Pe},\quad N=\frac{4\sigma T_w^3}{kK_a}$$

$$\tau_b = K_a b,\quad G^* = \frac{G}{4\sigma T_w^4}. \qquad (4)$$

The energy equation (1) and the equation of radiative transfer (3) may be transformed to the coupled system of partial differential equations

$$U\frac{\partial\phi}{\partial X} = \frac{\partial^2\phi}{\partial Y^2}+\frac{\partial^2\phi}{\partial Z^2}+N\tau_b^2(G-\phi^4) \qquad (5)$$

and

$$\frac{1}{Pe^2}\frac{\partial^2 G^*}{\partial X^2}+\frac{\partial^2 G^*}{\partial Y^2}+\frac{\partial^2 G^*}{\partial Z^2} = 3\tau_b^2(G^*-\phi^4) \qquad (6)$$

respectively.

With regards to the contribution of the first term on the left-hand side of equation (6), Echigo *et al.* [5] performed a series of numerical experiments for a conjugate version of this problem related to circular pipes. Their results showed that axial radiation penetrated one or two diameters in the upstream region of the heat exchange section. This conclusion was also verified by Pearce and Emery [4] using a more formal order-of-magnitude analysis for circular pipes. They proposed a radiative Peclet number that serves to define the threshold of axial radiation effects. That is, in terms of the hydraulic diameter, $Re_{D_h}\,Pr(\tau_b/N) > 10$.

Therefore, applying this criterion to equation (6), the axial variation of G^* may be dropped resulting in the two-dimensional equation applicable at the duct

$$\frac{\partial^2 G^*}{\partial Y^2}+\frac{\partial^2 G}{\partial Z^2} - 3\tau_b^2(G-\phi^4). \qquad (7)$$

To complete the formulation, the dimensionless boundary conditions imposed on equations (5) and (7) may be written in compact form as follows:

(a) at the entrance $X = 0$, $\phi = \phi_e$; (8)

(b) at the duct surface, $\phi = 1$; (9)

$$\frac{\partial G^*}{\partial Y} = -\tfrac{3}{2}\lambda_w \tau_b (G^* - 1) \quad \text{and} \quad \frac{\partial G^*}{\partial Z} = -\tfrac{3}{2}\lambda_w \tau_b (G^* - 1) \tag{10}$$

where

$$\lambda_w = \frac{\varepsilon_w}{2 - \varepsilon_w}. \tag{11}$$

3. PARAMETERS OF INTEREST

From a conceptual point of view, the mean bulk temperature is the most important thermal quantity in the analysis of forced convection in isothermal ducts. Its definition is given by

$$\phi_b(X) = \frac{\int_A U\phi \, dA}{\int_A U \, dA}. \tag{12}$$

An indirect way of calculating the total heat transfer rate Q_T necessitates the definition of the total Nusselt number, namely

$$Nu_T = \frac{q_{wt} D_h}{k(T_w - T_b)} \tag{13}$$

where q_{wT} denotes the local surface heat flux

$$q_{wT} = q_{wC} + q_{wR} \tag{14}$$

and q_{wC} and q_{wR} represent the wall heat flux due to conduction and radiation, respectively. This relation leads to the general expression

$$Nu_T = -\frac{\ln(\phi_b)}{4X}. \tag{15}$$

At this stage, it should be stressed that calculation of the total heat transfer involving the total Nusselt number is more elaborate. Nevertheless, these calculations were carried out with the purpose of comparing the numerical results with asymptotic solutions published in the literature.

4. NUMERICAL PROCEDURE

In spite of the fact that the system of partial differential equations (5) and (7) is amenable to a finite element or a finite difference treatment, we have implemented here a variant of the former; the method of lines (MOL), for the analysis of equation (5). Conversely, to be consistent with the discretization procedure adopted here, equation (7) may be solved by finite differences, also. This leads to an associated system of algebraic equations

$$\frac{G^*_{i-1,j} - 2G^*_{i,j} + G^*_{i+1,j}}{(\Delta Y)^2} + \frac{G^*_{i,j-1} - 2G^*_{i,j} + G^*_{i,j+1}}{(\Delta Z)^2} = g(G^*_{i,j})$$

$$i = 2, \ldots, I; \quad j = 2, \ldots, J. \tag{16}$$

The numerical solution of this system may be carried out at each axial station X using an adaptation of the Gaussian elimination algorithm for the numerical determination of G^* at each line.

5. TEST CASES

5.1. *Rectangular duct*

The dimensionless energy equation for a rectangular duct is

$$\frac{\partial \phi}{\partial X} = \frac{1}{U}\left(\frac{\partial^2 \phi}{\partial Y^2} + \frac{\partial^2 \phi}{\partial Z^2}\right) + \frac{N\tau_b^2}{U}(G^* - \phi^4) \tag{17}$$

where the appropriate velocity distribution U is taken from [11]

$$U = \left(\frac{m+1}{m}\right)\left(\frac{n+1}{n}\right)[1 - Y^n][1 - Z^m]$$

in which

$$m = 1.7 + 0.5(\alpha^*)^{-1.4}$$

$$n = \begin{cases} 2 & \text{for } \alpha^* \leqslant \frac{1}{3} \\ 2 + 0.3(\alpha^* - \frac{1}{3}) & \text{for } \alpha^* \geqslant \frac{1}{3}. \end{cases} \tag{18}$$

Equations (6) and (17) will be solved subjected to the boundary conditions of equations (8)–(11).

5.2. *Equilateral triangular duct*

The fully developed velocity distribution in an equilateral triangular duct can be written as [11]

$$U = \frac{15}{8}\left[(Y)^3 - 3(Y)(Z)^2 - 2(Y)^2 - 2(Z)^2 + \frac{32}{27}\right]. \tag{19}$$

The boundary conditions for the irradiation, G^*, on the hypotenuse will be

$$\frac{\partial G^*}{\partial n} = -\tfrac{3}{2}\lambda_w \tau_b (G^* - 1) \tag{20}$$

where n is the normal direction of the boundary.

6. DISCUSSION OF NUMERICAL RESULTS

A numerical investigation involving the simultaneous effect of convection and radiation in a non-circular duct having an isothermal wall has been conducted. The problem considered in this study contains several parameters, optical thickness, τ_b, conduction–radiation, N, wall emissivity, ε_w, dimensionless axial distance, X, and total Nusselt number, Nu_T. Numerical solutions were generated for a variety of combinations of the above parameters. All computations were performed in double precision on a VAX 8800. The total Nusselt number, equation (15), has been calculated numerically in the thermal entrance region of square, rectangular and equilateral triangular ducts, using the MOL. These have been presented in Figs. 1–3. For combined convection and radiation, $\phi = 0.5$ (the entrance-to-wall temperature ratio) is used throughout the calculation. The following sections are devoted to the discussion of each geometry.

6.1. *Rectangular duct*

Due to symmetry, the calculations are performed in 1/4 of the duct cross section utilizing 16 lines. The aspect ratios of $\alpha^* = 0.2$, 0.5 and 1.0 are chosen for the calculations related to this geometry. Figure 1 represents the total Nusselt number with a radiation–conduction parameter of $N = 3$ and an optical thickness of $\tau_b = 5$ for emissivities $\varepsilon_w = 0.1$ and 1.0. Inspection of the curves in this figure indicates that the Nusselt number is largest initially at the emissivity of $\varepsilon_w = 1.0$ and rectangular cross-section of $\alpha^* = 0.2$ and starts gradually decreasing until it reaches a minimum value and then gradually starts to increase again. The same behavior is observed with different aspect ratio and different emissivity.

Figure 2 shows the total Nusselt number with an emissivity of $\varepsilon_w = 1.0$, optical thickness of $\tau_b = 1.0$ for radiation–conduction parameters of $N = 1$ and 3. The curves in Fig. 2 indicate the same behavior as explained for Fig. 1.

The total Nusselt numbers with emissivity of $\varepsilon_w = 1.0$, radiation–conduction parameter of $N = 3$ for optical thickness of $\tau_b = 1.0$ and 5.0 are shown in Fig. 3. This figure indicates that variation in optical thickness has a great

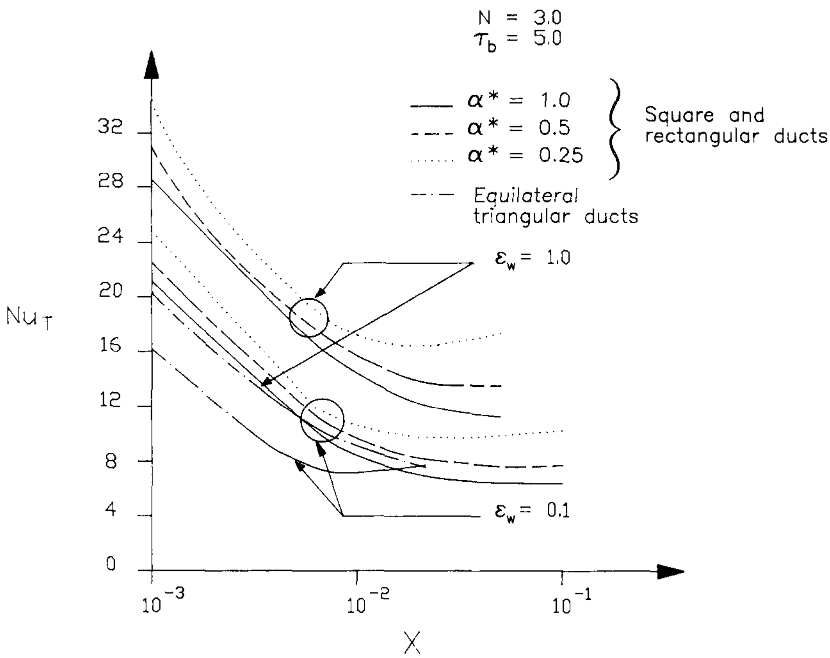

FIG. 1. Effect of wall emissivity on the total Nusselt number for rectangular and equilateral triangular ducts.

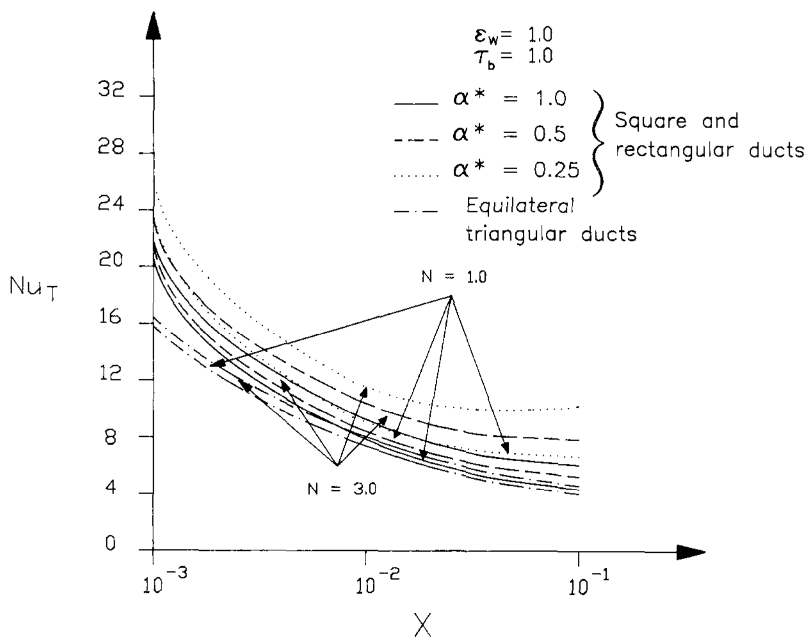

FIG. 2. Effect of radiation–conduction parameter on the total Nusselt number for rectangular and equilateral triangular ducts.

impact on the convection Nusselt number. At low optical thickness, the maximum radiation Nusselt number occurs at the end of the pipe as expected. At high values of optical thickness, the radiation Nusselt number becomes an increasing function of the axial.

6.2. *Equilateral triangular duct*

In this geometry, because of the symmetry, only 1/6 of the total area is used in the calculation domain. The results for this configuration are based on ten lines. Figure 1 represents the total Nusselt number with a radiation–conduction par-

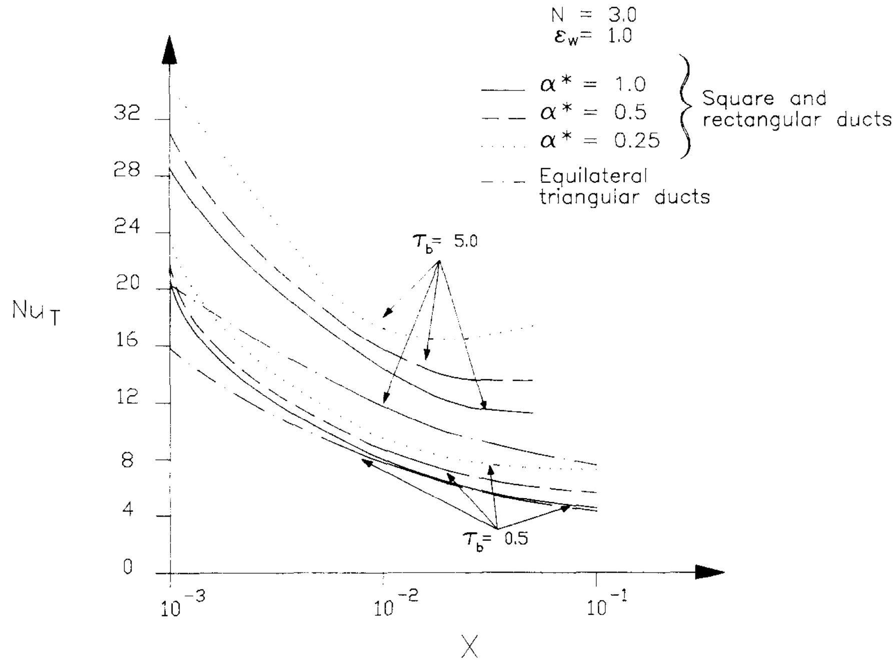

FIG. 3. Effect of optical thickness on the total Nusselt number for rectangular and equilateral triangular ducts.

ameter of $N = 3$ and an optical thickness of $\tau_b = 5$ for wall emissivities $\varepsilon_w = 0.1$ and 1.0. The resulting increase in the convective heat transfer with reduced emission is indicated in Fig. 1. Note that the emissivity at the entrance tends to accelerate the development of the thermal uniformity in the region.

Figure 2 represents total Nusselt number with a wall emissivity of $\varepsilon_w = 1.0$ and an optical thickness of $\tau_b = 1.0$ for radiation–conduction parameters of $N = 1$ and 3. It is evident from this figure that radiation considerably enhances the total heat transfer.

Figure 3 represents the total Nusselt number with a wall emissivity of $\varepsilon_w = 1$, radiation conduction parameter of $N = 3$ for an optical thickness of $\tau_b = 1$ and 5. The comparison of the set of curves for the total Nusselt number, Nu_T, in Fig. 3 indicates that for $\tau_b = 5$, Nu_T increases with X, always.

7. CONCLUSION

In this paper, the effect of the combined convective–radiative heat transfer in thermally developing gas flow of rectangular and equilateral triangular ducts was investigated. The resulting set of simultaneous integro-partial differential equations was solved numerically using the method of moments. An explicit finite-difference procedure for laminar gas flow which takes advantage of the method of lines was developed. The numerical scheme was found to be simple, accurate and efficient. It is ideally suited for the combined mode of heat transfer. Results of these calculations for pure convection agree very well with previous investigations. The effects of the different physical parameters were systematically studied. From the cases studied, the conclusions are that the influence of the radiation–conduction parameter on the thermal characteristic of the medium is substantial.

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